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On the unitary equivalence of absolutely continuous parts of self-adjoint extensions

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Abstract

The classical Weyl–von Neumann theorem states that for any self-adjoint operator A_0 in a separable Hilbert space $\mathfrak{H}$ there exists a (non-unique) Hilbert–Schmidt operator $C = C^*$ such that the perturbed operator $A_0 + C$ has purely point spectrum. We are interesting whether this result remains valid for non-additive perturbations by considering the set Ext_A of self-adjoint extensions of a given densely defined symmetric operator A in $\mathfrak{H}$ and some fixed $A_0 = A_0^* \in \mathrm{Ext}_A$. We show that the *ac*-parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ and A_0 are unitarily equivalent provided that the resolvent difference $K_{\widetilde{A}} := (\widetilde{A} - i)^{-1} - (A_0 - i)^{-1}$ is compact and the Weyl function $M(\cdot)$ of the pair $\{A, A_0\}$ admits weak boundary limits $M(t) := \text{w-lim}_{y\to+0} M(t + iy)$ for a.e. $t \in \mathbb{R}$. This result generalizes the classical Kato–Rosenblum theorem. Moreover, it demonstrates that for such pairs $\{A, A_0\}$ the Weyl–von Neumann theorem is in general not true in the class Ext_A.

Keywords: Symmetric operators; Self-adjoint extensions; Boundary triplets; Weyl functions; Unitary equivalence

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1. Introduction

Let A_0 be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $C = C^*$ be a trace class operator in $\mathfrak{H}$, $C \in \mathfrak{S}_1(\mathfrak{H})$. Recall, that according to the Kato–Rosenblum theorem, cf. [19,30] the absolutely continuous parts A_0^{ac} and $\widetilde{A}^{ac}$, in short the *ac*-parts, of the operators A_0 and $\widetilde{A} = A_0 + C$ are unitarily equivalent. In other words, the absolutely continuous spectrum, in short *ac*-spectrum, of A_0 and the spectral multiplicity function $N_{A_0^{ac}}(\cdot)$ of A_0^{ac} are stable under *additive* trace class perturbations. At the same time, the Weyl–von Neumann–Kuroda theorem [1, Theorem 94.2], [31,23,24] shows that the condition $C \in \mathfrak{S}_1(\mathfrak{H})$ cannot be replaced by $C \in \mathfrak{S}_p(\mathfrak{H})$ with $p \in (1, \infty]$ (where $\mathfrak{S}_p(\mathfrak{H})$ denotes the Neumann–Schatten operator ideals).

Theorem 1.1. *(See [20, Theorems 10.2.1 and 10.2.3].) For any operator $A_0 = A_0^*$ in $\mathfrak{H}$ and any $p \in (1, \infty]$ there exists an operator $C = C^* \in \mathfrak{S}_p(\mathfrak{H})$ such that the perturbed operator $\widetilde{A} = A_0 + C$ has purely point spectrum. In particular, $\sigma_{ac}(A_0 + C) = \emptyset$.*

The Kato–Rosenblum theorem was generalized by Birman [4] and Birman and Krein [6] to the case of *non-additive* perturbations. Namely, it was shown that A_0^{ac} and $\widetilde{A}^{ac}$ still remain unitary equivalent whenever

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_1(\mathfrak{H}).$$

In particular, this is true if A_0 and $\widetilde{A}$ are self-adjoint extensions of a symmetric operator A (in short $A_0, \widetilde{A} \in \mathrm{Ext}_A$). This rises the following Weyl–von Neumann problem for extensions: Given $p \in (1, \infty]$ and a self-adjoint extension A_0 of A. Does there exist a self-adjoint extension $\widetilde{A}$ of A such that $\widetilde{A}$ has purely point spectrum and the difference $(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1}$ belongs to $\mathfrak{S}_p(\mathfrak{H})$? To the best of our knowledge this problem was not investigated.

In the present paper we show that the Weyl–von Neumann theorem for extensions becomes false in general. We show that under an additional assumption on the symmetric operator A the *ac*-part of a certain extension $A_0 = A_0^*$ is unitarily equivalent to the *ac*-part of any extension $\widetilde{A} = \widetilde{A}^*$ of A provided that their resolvent difference is compact, that is,

$$K_{\widetilde{A}} := (\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{1.1}$$

The additional assumption on the pair $\{A, A_0\}$ is formulated in terms of the Weyl function of the pair $\{A, A_0\}$. The latter is the main object in the boundary triplet approach to the extension theory extensively developed in the last three decades, see [11,12,17] and references therein.

The core of this approach is the following abstract version of Green's formula

$$\big(A^* f, g\big) - \big(f, A^* g\big) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}\big(A^*\big), \tag{1.2}$$

where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ are linear mappings. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the operator A^* if (1.2) holds and the mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : \mathrm{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

With a boundary triplet Π for A^* one associates in a natural way the Weyl function $M(\cdot) = M_\Pi(\cdot)$ (see Definition 2.11), which is the key object of this approach. It is an operator-valued Nevanlinna function with values in $[\mathcal{H}]$ and its role in the extension theory is similar to that of the classical Weyl–Titchmarsh function in the spectral theory of Sturm–Liouville operators.

In particular, if A is simple, then $M(\cdot)$ determines the pair $\{A, A_0\}$, where $A_0 := A^* \upharpoonright \ker \Gamma_0$, uniquely, up to unitary equivalence. Moreover, $M(\cdot)$ is regular (holomorphic) precisely on the resolvent set $\varrho(A_0)$ of A_0 and the spectral properties of A_0 are described in terms of the limits $M(t + i0)$ at the real line (see [8]).

Our main result (Theorem 4.3) reads now as follows.

Theorem 1.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that the corresponding Weyl function $M(\cdot)$ admits weak limits*

$$M(t + i0) := \operatorname*{w-lim}_{y\downarrow 0} M(t + iy) \quad \textit{for a.e. } t \in \mathbb{R}. \tag{1.3}$$

If a self-adjoint extension $\widetilde{A}$ of A satisfies condition (1.1)*, then the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, respectively, are unitarily equivalent.*

We also present a certain local version of this result (cf. Corollary 4.6). Namely, we show that if the condition (1.3) holds for a.e. t of a measurable subset $\mathcal{D}$ of $\mathbb{R}$, then the corresponding parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{D})$ and $A^{ac} E_{A_0}(\mathcal{D})$ are unitarily equivalent provided that condition (1.1) is satisfied. Here $E_{\widetilde{A}}(\cdot)$ and $E_{A_0}(\cdot)$ stand for the spectral measures of $\widetilde{A}$ and A_0, respectively.

The condition (1.3) is independent from a choice of a boundary triplet. Moreover, it is rather strong. For instance, there exist operators for which the only condition (1.3) (without the compactness assumption (1.1)) yields ac-minimality of A_0^{ac}. The latter means that A_0^{ac} is contained in (i.e. is unitarily equivalent to a part of) any self-adjoint extension $\widetilde{A}$ of A. In particular, this effect takes place for some Schrödinger operators in the half-spaces (see [28] and Section 5 below). We plan to discuss this problem for elliptic operators in general unbounded domains in a separate publication.

The paper is organized as follows. In Section 2 we give a short introduction into the theory of ordinary and generalized boundary triplets and the corresponding Weyl functions. In Section 3 we express the spectral multiplicity function of the ac-part $\widetilde{A}^{ac}$ of $\widetilde{A} = \widetilde{A}^* (\in \mathrm{Ext}_A)$ by means of the corresponding Weyl function. Here we substantially use the multiplicity theory for non-orthogonal operator-valued measures on $\mathbb{R}$ developed in [27]. In Section 4 we apply this technique for proving Theorem 1.2. Moreover, we present a simple independent proof of the Kato–Rosenblum theorem without using a concept of the wave operators. Finally, Section 5 contains a short description of applications of Theorem 1.2 as well as condition (1.3) itself to Schrödinger operators which will be discussed in a forthcoming paper.

The main results of the paper have been announced (without proofs) in [29], a preliminary version with applications to elliptic operators in half-space has been published as a preprint [28].

Notations. We consider only separable Hilbert spaces which are denoted by $\mathfrak{H}$, $\mathcal{H}$ etc. The symbols $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $[\mathfrak{H}_1, \mathfrak{H}_2]$ stand for the set of closed densely defined linear operators and the set of bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, respectively. We set $\mathcal{C}(\mathcal{H}) := \mathcal{C}(\mathcal{H}, \mathcal{H})$ and $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. The symbols $\mathrm{dom}(\cdot)$, $\mathrm{ran}(\cdot)$, $\varrho(T)$ and $\sigma(T)$ denote the domain, the range, the resolvent set and the spectrum of an operator $T \in \mathcal{C}(\mathcal{H})$, respectively; T^{ac} and $\sigma_{ac}(T)$ stand for the ac-part and the ac-spectrum of an operator $T = T^* \in \mathcal{C}(\mathcal{H})$; E_T and $N_T(\cdot)$ denote the resolution of the identity and the multiplicity function of $T = T^* \in \mathcal{C}(\mathcal{H})$, respectively. For operator-valued measures Σ the multiplicity function is denoted by $N_\Sigma(t)$. If $E_T(\cdot)$ is the orthogonal spectral measure associated with a self-adjoint operator T, then we usually write $N_T(t)$ instead of $N_{E_T}(t)$.

$\mathfrak{S}_p(\mathfrak{H})$, $p \in [1, \infty]$, stand for the Schatten–von Neumann ideals in $\mathfrak{H}$. Denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of the line $\mathbb{R}$ and by $\mathcal{B}_b(\mathbb{R})$ the algebra of bounded subsets in $\mathcal{B}_b(\mathbb{R})$. The Lebesgue measure of a set $\delta \in \mathcal{B}(\mathbb{R})$ is denoted by $|\delta|$.

2. Preliminaries

2.1. Operator measures

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space. A mapping $\Sigma(\cdot) : \mathcal{B}_b(\mathbb{R}) \to [\mathcal{H}]$ is called an operator (operator-valued) measure if

(i) $\Sigma(\cdot)$ is σ-additive in the strong sense and
(ii) $\Sigma(\delta) = \Sigma(\delta)^* \geqslant 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$.

The operator measure $\Sigma(\cdot)$ is called bounded if it extends to the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, i.e. $\Sigma(\mathbb{R}) \in [\mathcal{H}]$. Otherwise, it is called unbounded. A bounded operator measure $\Sigma(\cdot) = E(\cdot)$ is called orthogonal if, in addition the conditions

(iii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$ for $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$ and $E(\mathbb{R}) = I_{\mathcal{H}}$

are satisfied.

Setting in (iii) $\delta_1 = \delta_2$, one gets that an orthogonal measure $E(\cdot)$ takes its values in the set of orthogonal projections on $\mathcal{H}$. Every orthogonal measure $E(\cdot)$ defines an operator $T = T^* = \int_{\mathbb{R}} \lambda \, dE(\lambda)$ in $\mathcal{H}$ with $E(\cdot)$ being its spectral measure. Conversely, by the spectral theorem, every operator $T = T^*$ in $\mathcal{H}$ admits the above representation with the orthogonal spectral measure $E =: E_T$.

By Σ^{ac}, Σ^s, Σ^{sc} and Σ^{pp} we denote absolutely continuous, singular, singular continuous and pure point parts of the measure Σ, respectively. The Lebesgue decomposition of Σ is given by $\Sigma = \Sigma^{ac} + \Sigma^s = \Sigma^{ac} + \Sigma^{sc} + \Sigma^{pp}$.

The operator measure Σ_1 is called subordinated to the operator measure Σ_2, in short $\Sigma_1 \prec \Sigma_2$, if $\Sigma_2(\delta) = 0$ yields $\Sigma_1(\delta) = 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$. If the measures Σ_1 and Σ_2 are mutually subordinated, then they are called equivalent, in short $\Sigma_1 \sim \Sigma_2$. Note, that there always exists a scalar measure ρ defined on $\mathcal{B}_b(\mathbb{R})$ such that $\Sigma \sim \rho$, see [27, Remark 2.2]. In particular, there always exists a scalar measure such that $\Sigma \prec \rho$.

Usually, with the operator-valued measure $\Sigma(\cdot)$ one associates a distribution operator-valued function $\Sigma(\cdot)$ defined by

$$\Sigma(t) = \begin{cases} \Sigma([0, t)), & t > 0, \\ 0, & t = 0, \\ -\Sigma([t, 0)), & t < 0, \end{cases} \tag{2.1}$$

which is called the spectral function of Σ. Clearly, $\Sigma(\cdot)$ is strongly left continuous, $\Sigma(t - 0) = \Sigma(t)$, and satisfies $\Sigma(t) = \Sigma(t)^*$, $\Sigma(s) \leqslant \Sigma(t)$, $s \leqslant t$.

Definition 2.2. (See [27, Definition 4.5].) Let Σ be an operator measure in $\mathcal{H}$ and let ρ be a scalar measure on $\mathcal{B}(\mathbb{R})$ such that $\Sigma \prec \rho$. Further, let $e = \{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Let

$$\Sigma_{ij}(t) := \big(\Sigma(t)e_i, e_j\big), \qquad \Psi_{ij}(t) := d\Sigma_{ij}(t)/d\rho,$$
$$\Psi_n^e(t) := \big(\Psi_{ij}(t)\big)_{i,j=1}^n, \qquad \Psi^e(t) := \big(\Psi_{ij}(t)\big)_{i,j=1}^\infty.$$

We call

$$N_\Sigma^e(t) := \operatorname{rank}\big(\Psi^e(t)\big) := \sup_{n\geqslant 1} \operatorname{rank}\big(\Psi_n^e(t)\big) \ \big(\operatorname{mod}(\rho)\big) \tag{2.2}$$

the multiplicity function.

By [27, Proposition 4.6] $N_\Sigma^e(\cdot)$ does not depend on the orthogonal basis e. Therefore one always has $N_\Sigma(t) := N_\Sigma^e(t)$ and one can omit the index e in (2.2). When applying this definition to the absolutely continuous part Σ^{ac} of Σ the scalar measure ρ^{ac} can be chosen to be the Lebesgue measure $|\cdot|$ on $\mathcal{B}(\mathbb{R})$.

The concept of the multiplicity function allows one to introduce the following definitions.

Definition 2.3. Let Σ_1 and Σ_2 be two operator measures.

(i) The operator measure Σ_1 is called spectrally subordinate to the operator measure Σ_2, in short $\Sigma_1 \prec\!\!\prec \Sigma_2$, if $\Sigma_1 \prec \Sigma_2$ and $N_{\Sigma_1}(t) \leqslant N_{\Sigma_2}(t)$ $(\operatorname{mod}(\Sigma_2))$.
(ii) The operator measures Σ_1 and Σ_2 are called spectrally equivalent, in short $\Sigma_1 \approx \Sigma_2$, if $\Sigma_1 \sim \Sigma_2$ and $N_{\Sigma_1}(t) = N_{\Sigma_2}(t)$ $(\operatorname{mod}(\Sigma_2))$.

In application to self-adjoint operators it makes sense to introduce the following definition.

Definition 2.4. Let $T_j = T_j^* \in \mathcal{C}(\mathfrak{H}_j)$, $j = 1, 2$. We say that T_1 is a part of T_2 if there is an isometry V from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ such that $VT_1V^* \subseteq T_2$.

Crucial for us in the sequel is the following theorem.

Theorem 2.5. *Let T_j be self-adjoint operators acting in $\mathfrak{H}_j$ with corresponding spectral measures $E_{T_j}(\cdot)$, $j = 1, 2$. Let $\mathcal{D} \in \mathcal{B}(\mathbb{R})$.*

(i) *$T_1E_{T_1}(\mathcal{D})$ is a part of $T_2E_{T_2}(\mathcal{D})$ if and only if $E_{T_1,\mathcal{D}} \prec\!\!\prec E_{T_2,\mathcal{D}}$, where $E_{T_j,\mathcal{D}}(\delta) := E_{T_j}(\delta \cap \mathcal{D})$, $j = 1, 2$.*
(ii) *The parts $T_1E_{T_1}(\mathcal{D})$ and $T_2E_{T_2}(\mathcal{D})$ are unitarily equivalent if and only if $E_{T_1,\mathcal{D}} \approx E_{T_2,\mathcal{D}}$.*

The proof follows immediately from [7, Theorem 7.5.1]. For $\mathcal{D} = \mathbb{R}$ Theorem 2.5 gives conditions for T_1 to be unitarily equivalent either to a part of T_2 or to T_2 itself.

If T_1 is a part of T_2, then $\sigma(T_1) \subseteq \sigma(T_2)$ and $N_{T_1}(t) \leqslant N_{T_2}(t)$ for a.e. $t \in \mathbb{R}$ $(\operatorname{mod}(E_{T_2}))$. Obviously, if T_1 is a part of T_2 and T_2 is a part of T_1, then T_1 and T_2 are unitarily equivalent.

Using Definition 2.4 Theorem 1.2 can be reformulated as follows: If the conditions (1.1) and (1.3) are satisfied, then A_0^{ac} and $\widetilde{A}^{ac}$ are parts of each other.

2.2. R-functions

Let $\mathcal{H}$ be a separable Hilbert space. We recall that an operator-valued function $F(\cdot)$ with values in $[\mathcal{H}]$ is called to be a Herglotz, Nevanlinna or R-function [1,3,17,22], if it is holomorphic in $\mathbb{C}_+$ and its imaginary part is non-negative, i.e. $\mathrm{Im}(F(z)) := (2i)^{-1}(F(z) - F(z)^*) \geqslant 0$, $z \in \mathbb{C}_+$. In what follows we prefer the notion of R-function. The class of R-functions with values in $[\mathcal{H}]$ will be denoted by $(R_{\mathcal{H}})$. Any $(R_{\mathcal{H}})$-function $F(\cdot)$ admits an integral representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma_F, \quad z \in \mathbb{C}_+ \tag{2.3}$$

(see, for instance, [1,3,22]), where $C_0 = C_0^*$, $C_1 \geqslant 0$ and Σ_F is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} (1+t^2)^{-1}\, d\Sigma_F \in [\mathcal{H}]$. The integral is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators the measure Σ_F is not necessarily orthogonal. However, the operator-valued measure Σ_F is uniquely determined by the R-function $F(\cdot)$. It is called the spectral measure of $F(\cdot)$. The associated spectral function is denoted by $\Sigma_F(t)$, $t \in \mathbb{R}$, cf. (2.1).

Let us calculate $N_{\Sigma_F^{ac}}(t)$, $t \in \mathbb{R}$. For any Hilbert–Schmidt operator $D \in \mathfrak{S}_2(\mathcal{H})$ satisfying $\ker(D) = \ker(D^*) = \{0\}$ let us consider the modified or sandwiched $R_{\mathcal{H}}$-function

$$\left(F^D\right)(z) := D^* F(z) D, \quad z \in \mathbb{C}_+.$$

For $F^D(\cdot)$ the strong limit $F^D(t) := F^D(t + i0) := \text{s-lim}_{y \to +\infty} F^D(t + iy)$ exists for a.e. $t \in \mathbb{R}$. We set

$$d_{F^D}(t) := \dim\left(\mathrm{ran}\left(\mathrm{Im}\left(F^D\right)(t)\right)\right), \quad \text{for a.e. } t \in \mathbb{R}. \tag{2.4}$$

Proposition 2.6. *Let* $F(\cdot) \in (R_{\mathcal{H}})$, $D \in \mathfrak{S}_2(\mathcal{H})$ *and* $\ker(D) = \ker(D^*) = \{0\}$. *Then* $N_{\Sigma_F^{ac}}(t) = d_{F^D}(t)$ *for a.e.* $t \in \mathbb{R}$.

Proof. It follows from (2.3) that

$$\mathrm{Im}\left(F(\lambda + iy)\right) = yC_1 + \int_{-\infty}^{\infty} \frac{y}{(t-\lambda)^2 + y^2}\, d\Sigma_F, \quad \lambda \in \mathbb{R}. \tag{2.5}$$

By Berezanskiĭ–Gel'fand–Kostyuchenko theorem [3,7] the derivative $\Psi_{D^*\Sigma_F D}(t) := \frac{d}{dt} D^* \Sigma_F(t) D$ exists for a.e. $t \in \mathbb{R}$ and the representation

$$D^* \Sigma_F^{ac}(\delta) D = \int_{\delta} \Psi_{D^*\Sigma_F D}(t)\, dt, \quad \delta \in \mathcal{B}_b(\mathbb{R})$$

holds. Applying the Fatou theorem (see [22]) to (2.5) and using (2.4) we obtain

$$\mathrm{Im}\left(\left(F^D\right)(\lambda)\right) = \pi \Psi_{D^*\Sigma_F D}(\lambda) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{2.6}$$

By [27, Corollary 4.7] $N_{\Sigma_F^{ac}}(\lambda) = \mathrm{rank}(\Psi_{D^*\Sigma_F D}(\lambda)) = \dim(\overline{\mathrm{ran}(\Psi_{D^*\Sigma_F D}(\lambda))})$ for a.e. $\lambda \in \mathbb{R}$. Finally, using (2.6) we get $N_{\Sigma_F^{ac}}(\lambda) = d_{F^D}(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. $\square$

Notice that Proposition 2.6 implies that $d_{F^D}(t)$ does not dependent on D. Assuming the existence of the limit $F(t) := \text{s-lim}_{y\to+0} F(t+iy)$ for a.e. $t \in \mathbb{R}$, we set

$$d_F(t) := \dim\bigl(\operatorname{ran}\bigl(\operatorname{Im}\bigl(F(t)\bigr)\bigr)\bigr)$$

for a.e. $t \in \mathbb{R}$. In this case Proposition 2.6 can be modified as follows.

Corollary 2.7. *Let $F(\cdot) \in (R_{\mathcal{H}})$. If the limit $F(t) := \text{s-lim}_{y\to+0} F(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{\Sigma_F^{ac}}(t) = d_F(t)$ for a.e. $t \in \mathbb{R}$.*

2.3. Boundary triplets and self-adjoint extensions

In this section we briefly recall the basic facts on boundary triplets and the corresponding Weyl functions, cf. [10–12,17].

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim(\ker(A^* \mp i)) \leqslant \infty$.

Definition 2.8. (See [17].) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$ are linear mappings, is called an (ordinary) boundary triplet for A^* if the "abstract Green's identity"

$$\bigl(A^* f, g\bigr) - \bigl(f, A^* g\bigr) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}\bigl(A^*\bigr), \tag{2.7}$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Definition 2.9. (See [17].) A closed extension A' of A is called a proper extension, in short $A' \in \operatorname{Ext}_A$, if $A \subset A' \subset A^*$.

Two proper extensions A', A'' are called disjoint if $\operatorname{dom}(A') \cap \operatorname{dom}(A'') = \operatorname{dom}(A)$ and transversal if in addition $\operatorname{dom}(A') + \operatorname{dom}(A'') = \operatorname{dom}(A^*)$.

Clearly, any self-adjoint extension $\widetilde{A} = \widetilde{A}^*$ is proper, $\widetilde{A} \in \operatorname{Ext}_A$. A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists whenever $n_+(A) = n_-(A)$. Moreover, the relations $n_\pm(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0) \cap \ker(\Gamma_1) = \operatorname{dom}(A)$ are valid. Besides, $\Gamma_0, \Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, where $\mathfrak{H}_+$ denotes the Hilbert space obtained by equipping $\operatorname{dom}(A^*)$ with the graph norm of A^*.

With any boundary triplet Π one associates two extensions $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j \in \{0, 1\}$, which are self-adjoint in view of Proposition 2.10 below. Conversely, for any extension $A_0 = A_0^* \in \operatorname{Ext}_A$ there exists a (non-unique) boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$.

Using the concept of boundary triplets one can parameterize all proper, in particular, self-adjoint extensions of A. For this purpose denote by $\widetilde{\mathcal{C}}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$, that is, the set of (closed) linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. The adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of a linear relation Θ in $\mathcal{H}$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : \bigl(h', k\bigr) = \bigl(h, k'\bigr) \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called *symmetric* if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

The multivalued part $\mathrm{mul}(\Theta)$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is $\mathrm{mul}(\Theta) = \{h \in \mathcal{H}\colon \{0, h\} \in \Theta\}$. Setting $\mathcal{H}_\infty := \mathrm{mul}(\Theta)$ and $\mathcal{H}_{\mathrm{op}} := \mathcal{H}_\infty^\perp$ we get $\mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathcal{H}_\infty$. This decomposition yields an orthogonal decomposition $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$ where $\Theta_\infty := \{0\} \oplus \mathrm{mul}(\Theta)$ and $\Theta_{\mathrm{op}} := \{\{f, g\} \in \Theta\colon\ f \in \mathrm{dom}(\Theta),\ g \perp \mathrm{mul}(\Theta)\}$. For the definition of the inverse and the resolvent set of a linear relation Θ we refer to [13].

Proposition 2.10. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\mathrm{Ext}_A \ni \widetilde{A} \to \Gamma\,\mathrm{dom}(\widetilde{A}) = \bigl\{\{\Gamma_0 f, \Gamma_1 f\}\colon\ f \in \mathrm{dom}(\widetilde{A})\bigr\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.8}$$

establishes a bijective correspondence between the sets Ext_A *and* $\widetilde{\mathcal{C}}(\mathcal{H})$. *We put $A_\Theta := \widetilde{A}$ where Θ is defined by* (2.8). *Moreover, the following hold*:

(i) $A_\Theta = A_\Theta^*$ *if and only if* $\Theta = \Theta^*$;
(ii) *The extensions A_Θ and A_0 are disjoint if and only if $\Theta \in \mathcal{C}(\mathcal{H})$. In this case* (2.8) *becomes*

$$A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0);$$

(iii) *The extensions A_Θ and A_0 are transversal if and only if $\Theta = \Theta^* \in [\mathcal{H}]$.*

In particular, $A_j := A^* \upharpoonright \ker(\Gamma_j) = A_{\Theta_j}$, $j \in \{0, 1\}$ where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$. Hence $A_j = A_j^*$ since $\Theta_j = \Theta_j^*$. In the sequel the extension A_0 is usually regarded as a reference self-adjoint extension.

2.4. Weyl functions and γ-fields

It is well known that Weyl functions give an important tool in the direct and inverse spectral theory of singular Sturm–Liouville operators. In [10–12] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A)$. Following [10–12] we recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.11. (See [10,11].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The functions $\gamma(\cdot) : \varrho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \varrho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \varrho(A_0), \tag{2.9}$$

$\mathfrak{N}_z := \ker(A^* - z)$, are called the *$\gamma$-field* and the *Weyl function*, respectively, corresponding to Π.

It follows from the identity $\mathrm{dom}(A^*) = \ker(\Gamma_0) \dotplus \mathfrak{N}_z$, $z \in \varrho(A_0)$, where $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, and $\mathfrak{N}_z := \ker(A^* - z)$, that the γ-field $\gamma(\cdot)$ is well defined and takes values in $[\mathcal{H}, \mathfrak{H}]$. Since $\Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, it follows from (2.9) that $M(\cdot)$ is well defined too and takes values in $[\mathcal{H}]$. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\varrho(A_0)$ and satisfy the following relations (see [11])

$$\gamma(z) = \bigl(I + (z - \zeta)(A_0 - z)^{-1}\bigr)\gamma(\zeta), \quad z, \zeta \in \varrho(A_0), \tag{2.10}$$

and

$$M(z) - M(\zeta)^* = (z - \bar{\zeta})\gamma(\zeta)^*\gamma(z), \quad z, \zeta \in \varrho(A_0). \tag{2.11}$$

The last identity yields that $M(\cdot)$ is an $R_{\mathcal{H}}$-function, that is, $M(\cdot)$ is an $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$M(z) = M(\bar{z})^* \quad \text{and} \quad \frac{\operatorname{Im}(M(z))}{\operatorname{Im}(z)} \geqslant 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, it follows from (2.11) that $M(\cdot)$ satisfies $0 \in \varrho(\operatorname{Im}(M(z)))$, $z \in \mathbb{C} \setminus \mathbb{R}$.

If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence (see [12,21]). Therefore $M(\cdot)$ contains (implicitly) full information on spectral properties of A_0. We recall that a symmetric operator is said to be *simple* if there is no non-trivial subspace which reduces it to a self-adjoint operator.

For a fixed extension $A_0 = A_0^*$ the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ satisfying $\operatorname{dom}(A_0) = \ker(\Gamma_0)$ is not unique. Let $\Pi_j = \{\mathcal{H}_j, \Gamma_0^j, \Gamma_1^j\}$, $j \in \{1, 2\}$, be two such triplets. Then the corresponding Weyl functions $M_1(\cdot)$ and $M_2(\cdot)$ are related by

$$M_2(z) = R^* M_1(z) R + R_0, \tag{2.12}$$

where $R_0 = R_0^* \in [\mathcal{H}_2]$ and $R \in [\mathcal{H}_2, \mathcal{H}_1]$ is boundedly invertible.

According to Proposition 2.10 the extensions A_Θ and A_0 are not disjoint whenever $\operatorname{mul}(\Theta) \neq \{0\}$. Considering A_Θ and A_0 as extensions of an intermediate extension $S := A_0 \upharpoonright (\operatorname{dom}(A_0) \cap \operatorname{dom}(A_\Theta))$ we can avoid this inconvenience.

Lemma 2.12. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$ its orthogonal decomposition. Further let $S := A_0 \upharpoonright (\operatorname{dom}(A_0) \cap \operatorname{dom}(A_\Theta))$. Then the triplet $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, defined by*

$$\widehat{\mathcal{H}} := \mathcal{H}_{\mathrm{op}} = \overline{\operatorname{dom}(\Theta)}, \qquad \widehat{\Gamma}_0 := \Gamma_0 \upharpoonright \operatorname{dom}\big(S^*\big), \qquad \widehat{\Gamma}_1 := \pi_{\mathrm{op}} \Gamma_1 \upharpoonright \operatorname{dom}\big(S^*\big),$$

is a boundary triplet for S^*, where π_{op} is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{op}}$, $A_0 = S^* \upharpoonright \ker(\widehat{\Gamma}_0)$ and $A_\Theta = S_{\Theta_{\mathrm{op}}}$. The corresponding Weyl function is

$$\widehat{M}(z) := \pi_{\mathrm{op}} M(z) \upharpoonright \mathcal{H}_{\mathrm{op}}, \quad z \in \mathbb{C}_\pm. \tag{2.13}$$

The proof can be found in [9]. Hence without loss of generality we can very often assume that the "coordinate" $\Theta := \Gamma \widetilde{A}$ of an extension $\widetilde{A} = A_\Theta = A_\Theta^* \in \operatorname{Ext}_A$ corresponds to the graph of a self-adjoint operator.

In what follows, without loss of generality, we always assume that the closed symmetric A is simple and, due to Lemma 2.12, the "coordinate" Θ of the extension $A_\Theta = A_\Theta^* \in \operatorname{Ext}_A$ is the graph of a self-adjoint operator.

2.5. Krein-type formula for resolvents and comparability

With any boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and any proper (not necessarily self-adjoint) extension $A_\Theta \in \operatorname{Ext}_A$ it is naturally associated the following (unique) Krein-type formula (cf. [10–12])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)\big(\Theta - M(z)\big)^{-1} \gamma(\bar{z})^*, \quad z \in \varrho(A_0) \cap \varrho(A_\Theta). \tag{2.14}$$

Formula (2.14) is a generalization of the well-known Krein formula for resolvents. We note also, that all objects in (2.14) are expressed in terms of the boundary triplet Π (cf. [10–12]). In other words, (2.14) gives a relation between Krein-type formula for canonical resolvents and the theory of abstract boundary value problems (framework of boundary triplets).

The following result is deduced from formula (2.14) (cf. [11, Theorem 2]).

Proposition 2.13. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta_i = \Theta_i^* \in \widetilde{\mathcal{C}}(\mathcal{H})$, $i \in \{1,2\}$. Then for any Schatten–von Neumann ideal $\mathfrak{S}_p$, $p \in (0,\infty]$, and any $z \in \mathbb{C} \setminus \mathbb{R}$ the following equivalence holds*

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \quad \Longleftrightarrow \quad (\Theta_1 - z)^{-1} - (\Theta_2 - z)^{-1} \in \mathfrak{S}_p(\mathcal{H}).$$

In particular, $(A_{\Theta_1} - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow (\Theta_1 - i)^{-1} \in \mathfrak{S}_p(\mathcal{H})$.

If in addition $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then for any $p \in (0,\infty]$ the equivalence holds

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \quad \Longleftrightarrow \quad \Theta_1 - \Theta_2 \in \mathfrak{S}_p(\mathcal{H}).$$

2.6. Generalized boundary triplets and proper extensions

In applications the concept of boundary triplets is too restrictive. Here we recall some facts on generalized boundary triplets following [12].

Definition 2.14. (See [12, Definition 6.1].) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triplet for A^** if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_j : \mathrm{dom}(\Gamma_j) \to \mathcal{H}$, $j = 0, 1$, are linear mappings such that $\mathrm{dom}(\Gamma) := \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1)$ is a core for A^*, Γ_0 is surjective, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ is self-adjoint and the following Green's formula holds

$$(A_* f, g) - (f, A_* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(A_*), \tag{2.15}$$

where $A_* := A^* \upharpoonright \mathrm{dom}(\Gamma)$.

By definition, $A_* := A^* \upharpoonright \mathrm{dom}(\Gamma)$ and $A_* \subseteq A^* = \overline{A_*}$ and $(A_*)^* = A$. Clearly, every ordinary boundary triplet is a generalized boundary triplet.

Lemma 2.15. *(See [12, Proposition 6.1].) Let A be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the following assertions are true*:

- (i) $\mathfrak{N}_z^* := \mathrm{dom}(A_*) \cap \mathbb{N}_z$ *is dense in* $\mathfrak{N}_z$ *and* $\mathrm{dom}(A_*) = \mathrm{dom}(A_0) + \mathfrak{N}_z^*$;
- (ii) $\overline{\Gamma_1 \,\mathrm{dom}(A_0)} = \mathcal{H}$;
- (iii) $\ker(\Gamma) = \mathrm{dom}(A)$ *and* $\overline{\mathrm{ran}(\Gamma)} = \mathcal{H} \oplus \mathcal{H}$.

Lemma 2.16. *Let A be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}^\top$ is closable and $\overline{\Gamma} \in \mathcal{C}(\mathfrak{H}_+, \mathcal{H})$.*

Proof. The Green's formula can be rewritten as $(A_* f, g) - (f, A_* g) = (J\Gamma f, \Gamma g)$ where $\Gamma := (\Gamma_0, \Gamma_1)^\top$ and $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Let $f_n \in \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1) = \mathrm{dom}(A_*)$, $\|f_n\|_{\mathfrak{H}_+} \to 0$ and $\Gamma f_n = \{\Gamma_0 f_n, \Gamma_1 f_n\} \to \{\varphi, \psi\}$ as $n \to \infty$. Hence

$$0 = \lim_{n\to\infty} \big[(A_* f_n, g) - (f_n, A_* g)\big] = (J f_\infty, \Gamma g), \quad \text{where } f_\infty := \{\varphi, \psi\}^\top.$$

Since $\mathrm{ran}(\Gamma)$ is dense in $\mathcal{H} \oplus \mathcal{H}$ one has $J f_\infty = 0$. Thus, $\varphi = \psi = 0$ and Γ is closable. □

For any generalized boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ we set $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j \in \{0, 1\}$. The extensions A_0 and A_1 are disjoint but not necessarily transversal. The latter holds if and only if Π is an ordinary boundary triplet. In general, the extension A_1 is only essentially self-adjoint.

Starting with Definition 2.14, one easily extends the definitions of γ-field and Weyl function to the case of a generalized boundary triplet Π by analogy with Definition 2.11 (cf. [12, Definition 6.2]).

Definition 2.17. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the operator-valued functions $\gamma(\cdot)$ and $M(\cdot)$ defined by

$$\gamma(z) := \big(\Gamma_0 \upharpoonright \mathfrak{N}_z^*\big)^{-1} : \mathcal{H} \to \mathfrak{N}_z \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \varrho(A_0), \tag{2.16}$$

are called the (generalized) *γ-field* and the *Weyl function* associated with the generalized boundary triplet Π, respectively.

It follows from Lemma 2.15(i) that $\gamma(\cdot)$ takes values in $[\mathcal{H}, \mathfrak{H}]$, $\mathrm{ran}(\gamma(z)) = \mathfrak{N}_z^* := \mathrm{dom}(A_*) \cap \mathfrak{N}_z$ and it satisfies the identity similar to that of (2.10) which shows that $\gamma(z)$ is a holomorphic operator-valued function on $\varrho(A_0)$.

Further, one has $\mathrm{dom}(M(z)) = \mathcal{H}$ since $\mathrm{ran}\,\gamma(z) \subset \mathrm{dom}(\Gamma_1)$, $z \in \varrho(A_0)$. By (2.16) $M(z)$ is closable since $\gamma(z)$ is bounded and Γ_1 is closable, by Lemma 2.16. Hence, by the closed graph theorem $M(\cdot)$ takes values in $[\mathcal{H}]$. Moreover, it is holomorphic on $\varrho(A_0)$, because so is $\gamma(\cdot)$, and satisfies the relation (2.11). It follows that $\ker(\mathrm{Im}\, M(z)) = \{0\}$, $z \in \mathbb{C}_+$, though the stronger condition $0 \in \varrho(\mathrm{Im}\, M(i))(\Longleftrightarrow \mathrm{ran}(\gamma(i)) = \mathfrak{N}_i)$ is satisfied if and only if Π is an ordinary boundary triplet (in the sense of Definition 2.8).

In the sequel we need the following simple but useful statement.

Proposition 2.18. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, $B = B^* \in \mathcal{C}(\mathcal{H})$ and $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. Let $\Gamma_1^B := \Gamma_0$ and $\Gamma_0^B := B\Gamma_0 - \Gamma_1$. Then*

(i) *$\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a generalized boundary triplet for A^* such that it holds* $\mathrm{dom}(A_*) := \mathrm{dom}(\Gamma) := \mathrm{dom}(A_0) + \mathrm{dom}(A_B) \subseteq \mathrm{dom}(A^*)$, *$A_*^* = A$;*
(ii) *the corresponding* (*generalized*) *Weyl function $M_B(\cdot)$ is*

$$M_B(z) = \big(B - M(z)\big)^{-1}, \quad z \in \mathbb{C}_\pm;$$

(iii) *Π_B is an* (*ordinary*) *boundary triplet if and only if $B = B^* \in [\mathcal{H}]$. In this case $M_B(\cdot)$ is an ordinary Weyl function in the sense of Definition* 2.8.

Note, an analogon of Proposition 2.10 does not hold for generalized boundary triplets. Nevertheless, since the corresponding Weyl function determines the pair $\{A, A_0\}$ uniquely, up to unitary equivalence, it is possible to describe the spectral properties of A_0 in terms of the (generalized) Weyl function $M(\cdot)$.

3. Weyl function and spectral multiplicity

Throughout of this section A is a densely defined simple closed symmetric operator in $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*, and let $M(\cdot)$ be the corresponding generalized Weyl function. Since $M(\cdot) \in (R_{\mathcal{H}})$ it admits representation (2.3). Since A is densely defined (see [12,26]), one gets $C_1 = 0$, i.e.

$$M(z) = C_0 + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Sigma_M.$$

Proposition 3.1. *Let A be a densely defined, simple closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $A_*(\subseteq A^*)$, $A_*^* = A$, and let $M(\cdot)$ be the corresponding Weyl function. If E_{A_0} is the spectral measure of $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, then $\Sigma_M \approx E_{A_0}$ and $\Sigma_M^{ac} \approx E_{A_0}^{ac}$.*

Proof. Alongside $\Sigma_M(\cdot)$ we introduce the bounded operator measure $\Sigma_M^0(\cdot)$,

$$\Sigma_M^0(\delta) = \int_\delta \frac{1}{1+t^2}\, d\Sigma_M, \quad \delta \in \mathcal{B}_b(\mathbb{R}).$$

Clearly, $\Sigma_M^0(\cdot) \approx \Sigma_M(\cdot)$. According to [2, formula (2.16)] one has

$$\Sigma_M^0(\delta) = \gamma(i)^* E_{A_0}(\delta)\gamma(i), \quad \delta \in \mathcal{B}(\mathbb{R}), \tag{3.1}$$

where $\gamma(\cdot)$ is the generalized γ-field of Π. Note, that though formula (3.1) is proved in [2] for ordinary boundary triplets, the proof remains valid for generalized boundary triplets. Due to the simplicity of A one has

$$\operatorname{span}\{(A_0 - z)^{-1} \operatorname{ran}(\gamma(i)) \colon z \in \mathbb{C}_+ \cup \mathbb{C}_-\} = \mathfrak{H}.$$

Hence the subspace $\mathfrak{N}_i := \overline{\mathfrak{N}_i^*}$, where $\mathfrak{N}_i^* := \operatorname{ran}(\gamma(i))$ is cyclic for A_0. Next, let P_i be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_i$. We set $\widetilde{\Sigma}_M^0(\cdot) := P_i E_{A_0}(\cdot) \upharpoonright \mathfrak{N}_i$.

Clearly, $\widetilde{\Sigma}_M^0(\cdot)$ is an operator measure. Since the linear manifold $\mathfrak{N}_i^*$ is cyclic for A_0, one gets from [27, Theorem 4.15] that the measures $\widetilde{\Sigma}_M^0$ and E_{A_0} are spectrally equivalent.

Note that $\Sigma_M^0(\cdot) = \gamma(i)^* \widetilde{\Sigma}_M^0(\cdot)\gamma(i)$. Since $\operatorname{ran}(\gamma(i))$ is dense in $\mathfrak{N}_i$, the latter yields $\Sigma_M^0 \sim \widetilde{\Sigma}_M^0$. Let $D \in \mathfrak{S}_2(\mathcal{H})$ and $\ker(D) = \ker(D^*) = \{0\}$. We set

$$\Psi_{D^*\Sigma_M^0 D}(t) := \frac{dD^*\Sigma_M^0(t)D}{d\rho(t)} \quad \text{and} \quad \Psi_{\widetilde{D}^*\widetilde{\Sigma}_M^0 \widetilde{D}}(t) := \frac{d\widetilde{D}^*\widetilde{\Sigma}_M^0(t)\widetilde{D}}{d\rho(t)}$$

where ρ is a scalar measure such that $\widetilde{\Sigma}_M^0 \sim \rho$ and $\widetilde{D} := \gamma(i)D : \mathcal{H} \to \mathfrak{N}_i$. We note that $\ker(\widetilde{D}) = \ker(\widetilde{D}^*) = \{0\}$. By [27, Corollary 4.7] we have

$$N_{\Sigma_M^0}(t) = \operatorname{rank}\bigl(\Psi_{D^*\Sigma_M^0 D}(t)\bigr) \quad \text{and} \quad N_{\widetilde{\Sigma}_M^0}(t) = \operatorname{rank}\bigl(\Psi_{\widetilde{D}^*\widetilde{\Sigma}_M^0\widetilde{D}}(t)\bigr)$$

for a.e. $t \in \mathbb{R}$ (mod(ρ)). Since $\Psi_{D^*\Sigma_M^0 D}(t) = \Psi_{\widetilde{D}^*\widetilde{\Sigma}_M^0\widetilde{D}}(t)$ for a.e. $t \in \mathbb{R}$ (mod(ρ)) we get $N_{\Sigma_M^0}(t) = N_{\widetilde{\Sigma}_M^0}(t)$ for a.e. $t \in \mathbb{R}$ (mod(ρ)). Hence $\widetilde{\Sigma}_M^0$ and Σ_M^0 are spectrally equivalent. Since $\widetilde{\Sigma}_M^0$ and E_{A_0} are spectrally equivalent, the measures Σ_M^0 and E_{A_0} are spectrally equivalent. This proves the first statement.

The second statement follows from the equality $\Sigma_M^{0,ac}(\delta) = \gamma(i)^* E_{A_0}^{ac}(\delta)\gamma(i)$, $\delta \in \mathcal{B}(\mathbb{R})$ where $\Sigma_M^{0,ac}$ is the ac-part of Σ_M^0. □

The proof of Proposition 3.1 leads to the following computing procedure for $N_{\Sigma_M^{ac}}(t)$: choosing $D \in \mathfrak{S}_2(\mathcal{H})$ such that $\ker(D) = \ker(D^*) = \{0\}$ we introduce the sandwiched Weyl function $M^D(\cdot)$,

$$\bigl(M^D\bigr)(z) := D^* M(z) D, \quad z \in \mathbb{C}_+.$$

It turns out that the limit $(M^D)(t) := \text{s-lim}_{y\to+0} M^D(t+iy)$ exists for a.e. $t \in \mathbb{R}$. We define in accordance with (2.13) the function $d_{M^D}(\cdot) : \mathbb{R} \to \mathbb{N} \cup \{\infty\}$,

$$d_{M^D}(t) := \operatorname{rank}\bigl(\operatorname{Im}\bigl(M^D(t)\bigr)\bigr) = \dim\bigl(\operatorname{ran}\bigl(\operatorname{Im}\bigl(M^D(t)\bigr)\bigr)\bigr),$$

which is well defined for a.e. $t \in \mathbb{R}$.

For a measurable non-negative function $\xi : \mathbb{R} \to \mathbb{R}_+$ defined for a.e. $t \in \mathbb{R}$ we introduce its support $\operatorname{supp}(\xi) := \{t \in \mathbb{R}:\ \xi(t) > 0\}$. By $\mathrm{cl}_{ac}(\cdot)$ we denote the absolutely continuous closure of a Borel set of $\mathbb{R}$, cf. Appendix A.

Proposition 3.2. *Let A be as in Proposition 3.1, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $A_*(\subseteq A^*)$, $A_*^* = A$, and let $M(\cdot)$ be the corresponding Weyl function. Further, let $E_{A_0}(\cdot)$ be the spectral measure of $A_0 = A_* \restriction \ker(\Gamma_0) = A_0^*$. If $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfies $\ker(D) = \ker(D^*) = \{0\}$, then $N_{A_0^{ac}}(t) = d_{M^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\operatorname{supp}(d_{M^D}))$.*

If, in addition, the limit $M(t) := \text{s-lim}_{y\to+0} M(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{A_0^{ac}}(t) = d_M(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\operatorname{supp}(d_M))$.

Proof. The relation $N_{A_0^{ac}}(t) = d_{M^D}(t)$ follows from Propositions 2.6 and 3.1. To prove $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\operatorname{supp}(d_{M^D}))$ we choose a total set $\{g_k\}_{k=1}^N$, $1 \leqslant N \leqslant \infty$, in $\mathcal{H}$. We set $h_k := Dg_k$. One easily verifies that $\{h_n\}_{n=1}^N$ is also a total set. We set $M_{h_n}(z) := (M(z)h_n, h_n)$, $z \in \mathbb{C}_+$. Clearly, $M_{h_n}(z)$ is R-function for every $n \in \{1, 2, \ldots, N\}$ and

$$M_{h_n}(t) := \lim_{y\to+0} M_{h_n}(t+iy) = \bigl(M(t)h_n, h_n\bigr)$$

exists for a.e. $t \in \mathbb{R}$. Set

$$\Omega_{ac}(M_{h_n}) := \bigl\{t \in \mathbb{R}:\ 0 < \operatorname{Im}\bigl(M_{h_n}(t)\bigr) < \infty\bigr\}.$$

Combining [8, Proposition 4.1] with Lemma A.3 we obtain

$$\sigma_{ac}(A_0) = \overline{\bigcup_{k=1}^{N} \mathrm{cl}_{ac}\big(\Omega_{ac}(M_{h_n})\big)} = \mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})\right). \tag{3.2}$$

If $t \in \mathrm{supp}(d_{M^D})$, then $\mathrm{Im}((M^D)(t)) \neq 0$. Hence $t \in \Omega_{ac}(M_{h_n})$ for some $n \in \{1, 2, \ldots, N\}$. Therefore $\mathrm{supp}(d_{M^D}) \subseteq \bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})$ which yields

$$\mathrm{cl}_{ac}\big(\mathrm{supp}(d_{M^D})\big) \subseteq \mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})\right). \tag{3.3}$$

Conversely, if $t \in \Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M^D}$, where $\mathcal{E}_{M^D} := \{t \in \mathbb{R}\colon \exists (M^D)(t)\}$, for some n, then $0 < d_{M^D}(t)$. Hence $\Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M^D} \subseteq \mathrm{supp}(d_{M^D})$ which yields $\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M^D} \subseteq \mathrm{supp}(d_{M^D})$. Hence

$$\mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M}\right) = \mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})\right) \subseteq \mathrm{cl}_{ac}\big(\mathrm{supp}(d_{M^D})\big).$$

Combining this equality with (3.2) and (3.3) we obtain $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}(d_{M^D}))$. □

Corollary 3.3. *Let A be as in Proposition* 3.2, *let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Further, let $B = B^* \in \mathcal{C}(\mathcal{H})$, $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $E_{A_B}(\cdot)$ the spectral measure of A_B. If $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfies* $\ker(D) = \ker(D^*) = \{0\}$, *then $N_{A_B^{ac}}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}(d_{M_B^D}))$.*

If, in addition, the limit $M_B(t) :=$ s-$\lim_{y\to+0} M_B(t + iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{A_B^{ac}}(t) = d_{M_B}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}(d_{M_B}))$.

Proof. By Proposition 2.18 $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a generalized boundary triplet for $A_* := A^* \upharpoonright \mathrm{dom}(A_*)$, $\mathrm{dom}(A_*) = \mathrm{dom}(A_0) + \mathrm{dom}(A_B)$, and $M_B(z) = (B - M(z))^{-1}$, $z \in \mathbb{C}_+$, is the corresponding generalized Weyl function. Clearly, $A_B = A_* \upharpoonright \ker(\Gamma_0^B)$. It remains to apply Proposition 3.2. □

This leads to the following theorem.

Theorem 3.4. *Let A be a densely defined closed symmetric operator, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Further, let $A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$, $B = B^* \in \mathcal{C}(\mathcal{H})$, and $E_{A_B}(\cdot)$ the spectral measure of A_B. Let $D \in \mathfrak{S}_2(\mathcal{H})$ and* $\ker(D) = \ker(D^*) = \{0\}$. *Then*:

(i) *$A_0 E_{A_0}^{ac}(\mathcal{D})$ is a part of $A_B E_{A_B}^{ac}(\mathcal{D})$ if and only if $d_{M^D}(t) \leqslant d_{M_B^D}(t)$ for a.e. $t \in \mathcal{D}$.*

(ii) *$A_0 E_{A_0}^{ac}(\mathcal{D})$ and $A_B E_{A_B}^{ac}(\mathcal{D})$ are unitarily equivalent if and only if $d_{M^D}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathcal{D}$.*

Proof. Without loss of generality we assume that A is simple since the self-adjoint part of A is contained as a direct summand in any self-adjoint extension of A. We show that $\Sigma_M^{ac}(\delta)=0$ for some $\delta\in\mathcal{B}_b(\mathbb{R})$ if and only if $d_{M^D}(t)=0$ for a.e. $t\in\delta$. By the Berezanskiĭ–Gel'fand–Kostyuchenko theorem [3,7] the derivative $\Psi_{D^*\Sigma_M D}(t):=\frac{d}{dt}D^*\Sigma(t)D$ exists and the relation

$$D^*\Sigma_M^{ac}(\delta\cap\mathcal{D})D=\int\limits_{\delta\cap\mathcal{D}}\Psi_{D^*\Sigma_M D}(t)\,dt,\quad \delta\in\mathcal{B}_b,$$

holds. One has $\Sigma_M^{ac}(\delta)=0$ if and only if $\Psi_{D^*\Sigma_M D}(t)=0$ for a.e. $t\in\delta$. Since $d_{M^D}(t)=\dim(\mathrm{ran}(\Psi_{D^*\Sigma_M D}(t)))$ for a.e. $t\in\mathbb{R}$ we find that $\Sigma_M^{ac}(\delta\cap\mathcal{D})=0$ if and only if $d_{M^D}(t)=0$ for a.e. $t\in\delta\cap\mathcal{D}$. Similarly we prove that $\Sigma_{M_B}^{ac}(\delta\cap\mathcal{D})=0$ if and only if $d_{D^*M_B D}(t)=0$ for a.e. $t\in\delta\cap\mathcal{D}$.

(i) Since by assumption $d_{M^D}(t)\leqslant d_{M_B^D}(t)$ for a.e. $t\in\mathcal{D}$, one gets by the considerations above that $\Sigma_M^{ac}(\delta\cap\mathcal{D})\prec\Sigma_{M_B}^{ac}(\delta\cap\mathcal{D})$. By Proposition 2.6 we have $N_{\Sigma_M^{ac}}(t)=d_{M^D}(t)$ and $N_{\Sigma_{M_B}^{ac}}(t)=d_{M_B^D}(t)$ for a.e. $t\in\mathbb{R}$. Hence $N_{\Sigma_M^{ac}}(t)\leqslant N_{\Sigma_{M_B}^{ac}}(t)$ for a.e. $t\in\mathcal{D}$ which proves that the restricted measures $\Sigma_M^{ac}(\cdot\cap\mathcal{D})$ are spectrally subordinated to $\Sigma_{M_B}^{ac}(\cdot\cap\mathcal{D})$, cf. Definition 2.3(i). Since $\Sigma_M^{ac}\approx E_{A_0}^{ac}$ and $\Sigma_{M_B}^{ac}\approx E_{A_B}^{ac}$, by Proposition 3.1, we get that $E_{A_0}^{ac}(\cdot\cap\mathcal{D})$ is spectrally subordinated to $E_{A_B}^{ac}(\cdot\cap\mathcal{D})$. Applying Theorem 2.5(i) we complete the proof.

(ii) If $d_{M^D}(t)=d_{D^*M_B D}(t)$ for a.e. $t\in\mathcal{D}$, then $\Sigma_M^{ac}(\cdot\cap\mathcal{D})\sim\Sigma_{M_B}^{ac}(\cdot\cap\mathcal{D})$. By Proposition 2.6, $N_{\Sigma_M^{ac}}(t)=d_{M^D}(t)$ and $N_{\Sigma_{M_B}^{ac}}(t)=d_{M_B^D}(t)$ for a.e. $t\in\mathbb{R}$ which implies that the operator measures $\Sigma_M^{ac}(\cdot\cap\mathcal{D})$ and $\Sigma_{M_B}^{ac}(\cdot\cap\mathcal{D})$ are spectrally equivalent, cf. Definition 2.3(ii). By Proposition 3.1, $E_{A_0}^{ac}(\cdot\cap\mathcal{D})$ and $E_{A_B}^{ac}(\cdot\cap\mathcal{D})$ are spectrally equivalent. Applying Theorem 2.5(ii) we prove that the absolutely continuous parts $A_0E_{A_0}^{ac}(\mathcal{D})$ and $A_BE_{A_B}^{ac}(\mathcal{D})$ are unitarily equivalent. $\square$

Theorem 3.4 reduces the problem of unitary equivalence of *ac*-parts of certain self-adjoint extensions of A to investigation of the functions $d_{M^D}(\cdot)$ and $d_{M_B^D}(\cdot)$.

Definition 3.5. Let A be a symmetric operator in $\mathfrak{H}$ and $A_0=A_0^*\in\mathrm{Ext}_A$. We say that A_0 is *ac-minimal* if A_0^{ac} is a part of any $\widetilde{A}=\widetilde{A}^*\in\mathrm{Ext}_A$.

In particular, if A_0 is *ac*-minimal, then $\sigma_{ac}(\widetilde{A})\supseteq\sigma_{ac}(A_0)$ and $N_{A_0^{ac}}(t)\leqslant N_{\widetilde{A}^{ac}}(t)$ for a.e. $t\in\mathbb{R}$ $(\mathrm{mod}(E_{\widetilde{A}^{ac}}))$ for any self-adjoint extension $\widetilde{A}\in\mathrm{Ext}_A$. Notice that an *ac*-minimal extension of A is not unique. However, the following corollary holds.

Corollary 3.6. *Let A be as in Theorem* 3.4. *If the self-adjoint extensions $\widetilde{A}$ and $\widetilde{A}'$ of A are ac-minimal, then their ac-parts are unitarily equivalent.*

4. Unitary equivalence

4.1. Preliminaries

In what follows we assume that A is a densely defined simple closed symmetric operator in $\mathfrak{H}$. By A_0 we denote a self-adjoint extension of A which is fixed. Alongside A_0 we consider $\widetilde{A}=\widetilde{A}^*\in\mathrm{Ext}_A$. It is known (see [11]) that there exists a boundary triplet $\Pi:=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* such that $A_0:=A^*\upharpoonright\ker(\Gamma_0)$. Of course, the boundary triplet Π is not uniquely determined by the assumption $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. If Π_1 and Π_2 are two such boundary triplets for A^*, then their Weyl functions $M_1(\cdot)$ and $M_2(\cdot)$ are related by (2.12) (cf. [11]).

Fix a boundary triplet $\Pi := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 = A^* \ker(\Gamma_0)$. By Proposition 2.10 there is a linear relation $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ such that $\widetilde{A} = A_\Theta$. In general, Θ is not the graph of an operator, $\Theta \notin \mathcal{C}(\mathcal{H})$. However, let us assume that Θ is the graph an operator B. By condition (1.1) and Proposition 2.13 we get that $(B - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, that means, that B is a self-adjoint operator with discrete spectrum. Hence, $\varrho(B) \cap \mathbb{R} \neq \emptyset$. In what follows we assume without loss of generality that $0 \in \varrho(B)$. According to the polar decomposition we have $B^{-1} = DJD$ where

$$D := |B|^{-1/2} = D^* \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \text{and} \quad J := \operatorname{sign}(B) = J^* = J^{-1}. \tag{4.1}$$

Clearly, $D \in \mathfrak{S}_\infty(\mathcal{H})$, $\ker(D) = \{0\}$, and D commutes with J. We set

$$G(z) := J - M^D(z), \quad z \in \mathbb{C}_+, \tag{4.2}$$

$M^D(z) := DM(z)D$, $z \in \mathbb{C}_+$, as usually. Obviously, $M^D(z)$ and $-G(z)$ are R-functions. Moreover, $\ker(G(z)) = \{0\}$ for any $z \in \mathbb{C}_+$. Indeed, if $G(z)f = 0$, then $Jf = DM(z)Df$. Hence, $\operatorname{Im}(M(z)Df, Df) = \operatorname{Im}(Jf, f) = 0$ which yields $Df = 0$ or $f = 0$. Since J is a Fredholm operator satisfying $\ker(J) = \ker(J^*) = \{0\}$ we find by [20, Theorem 5.26] that $G(z)$ is boundedly invertible for $z \in \mathbb{C}_+$. We set $T(z) := G(z)^{-1}$, $z \in \mathbb{C}_+$ and note that $T(\cdot)$ is a Nevanlinna function because so is $M^D(\cdot)$. Moreover, $T(z) - J = T(z)M^D(z)J \in \mathfrak{S}_\infty(\mathfrak{H})$ for $z \in \mathbb{C}_+$.

4.2. Trace class perturbations: Rosenblum–Kato theorem

Here we apply the Weyl function technique in order to obtain a simple and quite different proof of the classical Rosenblum–Kato theorem. In fact, we prove a generalization of the Rosenblum–Kato theorem due to Birman and Krein [6] which includes non-additive (trace class) perturbations. Our proof demonstrates the main idea of the proof of more general results contained in the next subsection.

Theorem 4.1. *Let A_0 and $\widetilde{A}$ be self-adjoint operators in $\mathfrak{H}$ satisfying*

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \tag{4.3}$$

Then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.

Proof. To include the operators $\widetilde{A}^{ac}$ and A_0^{ac} in the framework of extension theory we set

$$A := A_0 \upharpoonright \operatorname{dom}(A), \qquad \operatorname{dom}(A) = \big\{ f \in \operatorname{dom}(\widetilde{A}) \cap \operatorname{dom}(A_0):\ A_0 f = \widetilde{A} f \big\}.$$

Obviously, we have $A := \widetilde{A} \upharpoonright \operatorname{dom}(A)$. Clearly, A is a closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and $A_0, \widetilde{A} \in \operatorname{Ext}_A$.

First we assume that A is densely defined. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an (ordinary) boundary triplet for A^*, such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, and $M(\cdot)$ the corresponding Weyl function. By definition, $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_A$ and $\widetilde{A}$ and A_0 are disjoint, that is, $\operatorname{dom}(A) = \operatorname{dom}(A_0) \cap \operatorname{dom}(\widetilde{A})$. Hence, by Proposition 2.10(ii), there exists an operator $B = B^* \in \mathcal{C}(\mathcal{H})$ such that $\widetilde{A} = A_B$.

It follows from (2.14) and (4.3) that $M_B(z) := (B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \mathbb{C}_+$. In accordance with [5, Lemma 2.4], see also [32], the limits $M_B(t) := \lim_{y\to+0} M_B(t+iy)$ exist in $\mathfrak{S}_2(\mathcal{H})$, for a.e. $t \in \mathbb{R}$. By Theorem 3.4 it suffices to calculate the multiplicity function $d_{M_B}(t) := \operatorname{rank}(M_B(t)) = \dim(\operatorname{ran}(\operatorname{Im}(M_B(t))))$.

It follows from (4.1) and (4.2) that

$$\begin{aligned} T(z) = G(z)^{-1} &= \bigl(J - M^D(z)\bigr)^{-1} = \bigl(J - DM(z)D\bigr)^{-1} \\ &= D^{-1}\bigl(D^{-1}JD^{-1} - M(z)\bigr)^{-1}D^{-1} = |B|^{1/2}\bigl(B - M(z)\bigr)^{-1}|B|^{1/2}, \end{aligned} \tag{4.4}$$

$z \in \mathbb{C}_+$. Combining this relation with (4.1) yields

$$M_B(z) := \bigl(B - M(z)\bigr)^{-1} = DT(z)D, \quad z \in \mathbb{C}_+.$$

In turn, this equality implies

$$\operatorname{Im}\bigl(M_B(z)\bigr) = DT(z)^* \operatorname{Im}\bigl(M^D(z)\bigr)T(z)D, \quad z \in \mathbb{C}_+. \tag{4.5}$$

Moreover, since $M^D(z) \in \mathfrak{S}_1(\mathcal{H})$ and $T(z) - J \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \mathbb{C}_+$, by [5, Lemma 2.4] (see also [32]), for a.e. $t \in \mathbb{R}$ and $y \to 0$ there exist the limits $M^D(t)$ and $T(t)$ in $\mathfrak{S}_2(\mathcal{H})$-norm of the $R_{\mathcal{H}}$-functions $M^D((t+iy))$ and $T(t+iy)$, respectively. Therefore passing to the limit in (4.5) as $y \to 0$ we get

$$\operatorname{Im}\bigl(M_B(t)\bigr) = DT(t)^* \operatorname{Im}\bigl(M^D(t)\bigr)T(t)D \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.6}$$

Therefore we find

$$\begin{aligned} d_{M_B}(t) &= \dim\bigl(\operatorname{ran}\bigl(\operatorname{Im}\bigl(M_B(t)\bigr)\bigr)\bigr) \\ &= \dim\Bigl(\operatorname{ran}\Bigl(\sqrt{\operatorname{Im}\bigl(M_B(t)\bigr)}\Bigr)\Bigr) = \dim\Bigl(\operatorname{ran}\Bigl(\sqrt{\operatorname{Im}\bigl(M^D(t)\bigr)}T(t)D\Bigr)\Bigr). \end{aligned} \tag{4.7}$$

Since $(J - M^D(t))T(t) = T(t)(J - M^D(t)) = I$ for a.e. $t \in \mathbb{R}$, we find $\operatorname{ran}(T(t)) = \mathcal{H}$ for a.e. $t \in \mathbb{R}$. Combining this relation with $\overline{\operatorname{ran}}(D) = \mathcal{H}$ and (4.7) we obtain

$$d_{M_B}(t) = \dim\Bigl(\operatorname{ran}\Bigl(\sqrt{\operatorname{Im}\bigl(M^D(t)\bigr)}\Bigr)\Bigr) = \dim\bigl(\operatorname{ran}\bigl(\operatorname{Im}\bigl(M^D(t)\bigr)\bigr)\bigr) = d_{M^D}(t) \tag{4.8}$$

for a.e. $t \in \mathbb{R}$. Applying Theorem 3.4(ii) we complete this part of the proof.

If A is not densely defined one can repeat the above reasonings applying only the boundary triplet technique for non-densely defined symmetric operators developed in [12,26]. It turns out that the proof above can easily be carried over to this case. □

In the following corollary we show that in proving of unitary equivalence of A_0 and $\widetilde{A} \in \operatorname{Ext}_A$ it suffices to restrict the consideration to the case of disjoint extensions.

Corollary 4.2. *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, and let $M(\cdot)$ be the corresponding Weyl function. Let also $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $\mathcal{D} \in \mathcal{B}(\mathbb{R})$.*

(i) *If* $A_0^{ac}E_{A_0}(\mathcal{D})$ *is a part of* $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ *for any extension* $\widetilde{A}=\widetilde{A}^*\in \mathrm{Ext}_A$ *disjoint with* A_0, *then* $A_0^{ac}E_{A_0}(\mathcal{D})$ *is a part of* $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ *for any extension* $\widetilde{A}=\widetilde{A}^*\in \mathrm{Ext}_A$.

(ii) *If* $A_0^{ac}E_{A_0}(\mathcal{D})$ *is unitarily equivalent to* $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ *for any extension* $\widetilde{A}=\widetilde{A}^*\in \mathrm{Ext}_A$ *disjoint with* A_0, *then* $A_0^{ac}E_{A_0}(\mathcal{D})$ *is unitarily equivalent to the absolutely continuous part* $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ *of any extension* $\widetilde{A}=\widetilde{A}^*\in \mathrm{Ext}_A$.

Proof. By Proposition 2.10 an extension $\widetilde{A}\in \mathrm{Ext}_A$ which is not disjoint with A_0 admits a representation $\widetilde{A}_\Theta$ with $\Theta=\Theta^*\in \widetilde{\mathcal{C}}(\mathcal{H})\setminus \mathcal{C}(\mathcal{H})$. However, Θ admits a decomposition $\mathcal{H}=\mathcal{H}_{\mathrm{op}}\oplus\mathcal{H}_\infty$, $\Theta=\Theta_{\mathrm{op}}\oplus\Theta_\infty$ where Θ_{op} is the graph of the operator $B_{\mathrm{op}}=B_{\mathrm{op}}^*\in\mathcal{C}(\mathcal{H}_{\mathrm{op}})$ (cf. Section 2). Denoting by π_{op} the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{op}}$ and $M_{\mathrm{op}}(z):=\pi_{\mathrm{op}}M(z)\upharpoonright\mathcal{H}_{\mathrm{op}}$, we get $(\Theta-M(z))^{-1}=(B_{\mathrm{op}}-M_{\mathrm{op}}(z))^{-1}\pi_{\mathrm{op}}$. Therefore formula (2.14) takes the form

$$(A_\Theta-z)^{-1}-(A_0-z)^{-1}=\gamma(z)\big(B_{\mathrm{op}}-M_{\mathrm{op}}(z)\big)^{-1}\pi_{\mathrm{op}}\gamma(\bar z)^*,\quad z\in\mathbb{C}_\pm.$$

Choose an operator $B_\infty=B_\infty^*\in\mathcal{C}(\mathcal{H}_\infty)$ such that $(B_\infty-i)^{-1}\in\mathfrak{S}_1(\mathcal{H}_\infty)$ and put $B=B_{\mathrm{op}}\oplus B_\infty$. It follows from Proposition 2.13 that

$$(A_\Theta-z)^{-1}-(A_B-z)^{-1}\in\mathfrak{S}_1(\mathfrak{H}),$$

since $(B_\infty-i)^{-1}\in\mathfrak{S}_1(\mathcal{H}_\infty)$. By Theorem 4.1 the absolutely continuous parts A_Θ^{ac} and A_B^{ac} of A_Θ and A_B, respectively, are unitarily equivalent.

(i) Since by assumption $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $A_B^{ac}E_{A_B}(\mathcal{D})$ and A_B^{ac} is unitarily equivalent to A_Θ^{ac} we get that $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $A_\Theta^{ac}E_{A_\Theta}(\mathcal{D})$.

(ii) Since, by assumption, $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to $A_B^{ac}E_{A_B}(\mathcal{D})$ and A_B^{ac} is unitarily equivalent to A_Θ, we get that $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to $A_\Theta^{ac}E_{A_\Theta}(\mathcal{D})$. □

4.3. Compact non-additive perturbations

Here we generalize the Rosenblum–Kato theorem for the case of compact perturbations. To this end we assume that the maximal normal function

$$m^+(t):=\sup_{0<y\leqslant 1}\big\|M(t+iy)\big\|$$

is finite for a.e. $t\in\mathbb{R}$. This is the case if and only if the normal limits $M(t):=\text{w-lim}_{y\to+0}M(t+iy)$ exist and are bounded operators for a.e. $t\in\mathbb{R}$. Indeed, let $D=D^*$ be a Hilbert–Schmidt operator such that $\ker(D)=\{0\}$ and let $M^D(z):=DM(z)D$, $z\in\mathbb{C}_+$. Since the limit $M^D(t):=\text{o-lim}_{y\to+0}M^D(t+iy)$ exists and is a bounded operator for a.e. $t\in\mathbb{R}$, see [5,32], we find that

$$\lim_{y\to+0}\big(M(t+iy)Df,Dg\big)=\big(M^D(t)f,g\big),\quad f,g\in\mathcal{H},\ \text{for a.e. } t\in\mathbb{R}.$$

Hence the limit $\lim_{y\to+0}(M(t+iy)h,k)$ exists for a.e. $t\in\mathbb{R}$ and $h,k\in\mathrm{ran}(D)$ which yields the existence of $M(t):=\text{w-lim}_{y\to+0}M(t+iy)$ for a.e. $t\in\mathbb{R}$. The converse statement is obvious.

Now we are ready to prove the main result of this section.

Theorem 4.3. *Let A be a densely defined, closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, and let $M(\cdot)$ be the corresponding Weyl function. Let $\widetilde{A}$ be a self-adjoint extension of A and $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. If the maximal normal function $m^+(t)$ is finite for a.e. $t \in \mathbb{R}$ and condition (1.1) is satisfied, then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.*

Proof. We divide the proof into several steps.

(i) First we assume that the extensions $\widetilde{A}$ and A_0 are disjoint, that is $\widetilde{A} = A_B$ where $B = B^* \in \mathcal{C}(\mathcal{H})$. We define the operator $D \in \mathfrak{S}_\infty(\mathcal{H})$ in accordance with (4.1), $D := |B|^{-1/2}$, and investigate the function $M^D(z) := M^D(z) := DM(z)D$, $z \in \mathbb{C}_+$. Let $M^D(t) := DM(t)D$. Since the (weak) limit $M(t) := \text{w-lim}_{y\to+0} M(t+iy)$ exists for a.e. $t \in \mathbb{R}$ and $D \in \mathfrak{S}_\infty$, by [16, Theorem 3.6.3] (see also [32, Lemma 6.1.4]), the following limit exists

$$\operatorname*{o-lim}_{y\to+0} \left\| M^D(t+iy) - M^D(t) \right\| = 0 \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.9}$$

Let $\delta_a := \{t \in \mathbb{R}\colon \|M(t)\| \leqslant a\}$. Since $D = D^*$ is a non-negative compact operator, it admits the spectral decomposition

$$D = \sum_{l\in\mathbb{N}} \mu_l Q_l$$

where $\{\mu_l\}_{l=1}^\infty$, is the decreasing sequence of its eigenvalues, $\{Q_l\}_{l\in\mathbb{N}}$ the corresponding sequence of eigenprojections, $\dim\{Q_l\} < \infty$.

Since $\mu_l \to 0$ as $l \to \infty$, there exists a number $l_a \in \mathbb{N}$ such that $\mu_{l_a} < 1/\sqrt{2a}$. We put $\mathcal{H}_1 := \bigoplus_{l=l_a+1}^\infty Q_l\mathcal{H}$ and $\mathcal{H}_2 := \bigoplus_{l=1}^{l_a} Q_l\mathcal{H}$. Clearly, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\dim(\mathcal{H}_2) < \infty$. Moreover, the operator D admits the following decomposition $D = D_1 \oplus D_2$ where

$$D_1 := \sum_{l=l_a+1}^{\infty} \mu_l Q_l \quad \text{and} \quad D_2 := \sum_{l=1}^{l_a} \mu_l Q_l.$$

Since $\mu_{l_a} < 1/\sqrt{2a}$, we have $\|D_1\| < 1/\sqrt{2a}$. Hence

$$\left\| D_1 M(t) D_1 \right\| < 1/2, \quad t \in \delta_a. \tag{4.10}$$

Denote by P_1 and P_2 the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Note that $P_1 J = J P_1$ and $P_2 J = J P_2$.

(ii) Our next aim is to show that the operator function $G(z) := J - M^D(z)$ is invertible in $\mathbb{C}_+$ and that $T(z) := G(z)^{-1}$ has the limits $T(t) := \text{s-lim}_{y\to+0} T(t+iy)$ for a.e. $t \in \delta_a$. For this purpose we consider the decompositions

$$M^D(z) := \left(D_i M(z) D_j\right)_{i,j=1}^2 := \begin{pmatrix} M_{11}^D(z) & M_{12}^D(z) \\ M_{21}^D(z) & M_{22}^D(z) \end{pmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{matrix} \to \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{matrix},$$

$z \in \mathbb{C}_+$, and

$$G(z) = J - M^D(z) = \begin{pmatrix} J_1 - M_{11}^D(z) & -M_{12}^D(z) \\ -M_{21}^D(z) & J_2 - M_{22}^D(z) \end{pmatrix}, \quad z \in \mathbb{C}_+,$$

where $J_1 := JP_1$ and $J_2 := JP_2$.

(ii)$_1$ Let us prove that $\ker(J_1 - M_{11}^D(z)) = \{0\}$ for $z \in \mathbb{C}_+$. Indeed, from $0 = J_1 g - M_{11}^D(z)g = J_1 g - D_1 M(z) D_1 g$ one gets that $0 = \mathrm{Im}(M_{11}^D(z)g, g) = (\mathrm{Im}(M(z))D_1 g, D_1 g)$. Hence $0 = D_1 g = Dg$ which yields $g = 0$. Since $0 \in \varrho(J_1)$ and $M_{11}^D(\cdot) \in \mathfrak{S}_\infty$, we obtain that the operator $J_1 - M_{11}^D(z) = J_1(I_1 - J_1 M_{11}^D(z))$ is boundedly invertible for every $z \in \mathbb{C}_+$. Since $M_{11}^D(z)$ is an $R_{\mathcal{H}_1}$-function, we get that $\Xi(z) := (J_1 - M_{11}^D(z))^{-1}$, $z \in \mathbb{C}_+$, is an $R_{\mathcal{H}_1}$-function too.

(ii)$_2$ We show that for a.e. $t \in \delta_a$, $a > 0$, the limit $\Xi(t) := \text{o-}\lim_{y\to+0} \Xi(t+iy)$ exists in the operator norm and the following representation holds

$$\Xi(t) = \big(J_1 - M_{11}^D(t)\big)^{-1}. \tag{4.11}$$

First we note that $J_1 - M_{11}^D(z) = J_1(I_1 - J_1 M_{11}^D(z))$. Using (4.10) we get $\|J_1 M_{11}^D(t)\| < 1$ for $t \in \delta_a$. Hence the inverse operator $(I_1 - J_1 M_{11}^D(t))^{-1}$ exists for $t \in \delta_a$. Using $(J_1 - M_{11}^D(t))^{-1} = (I_1 - J_1 M_{11}^D(t))^{-1} J_1$ we find that the inverse operator $(J_1 - M_{11}^D(t))^{-1}$ exists for $t \in \delta_a$. Since $M_{11}^D(z)$ has limits $M_{11}^D(t)$ for a.e. $t \in \mathbb{R}$ one gets that $J_1 M_{11}^D(t) = \text{o-}\lim_{y\to+0} J_1 M_{11}^D(t+iy)$ for a.e. $t \in \mathbb{R}$. Fix any such $t_0 \in \delta_a$. Then due to estimate (4.10) there exists $\eta = \eta(t_0)$ such that $\sup_{y\in(0,\eta)} \|J_1 M_{11}^D(t_0+iy)\| \leqslant 1/2$. Therefore, the family $\{\|(I_1 - J_1 M_{11}^D(t_0+iy))^{-1}\|\}_{y\in(0,\eta)}$ is uniformly bounded for any fixed $t_0 \in \delta_a$. Using this fact and (4.9) we can pass to the limit as $y \to 0$ in the identity

$$\begin{aligned} &\big(I_1 - J_1 M_{11}^D(t_0+iy)\big)^{-1} - \big(I_1 - J_1 M_{11}^D(t_0)\big)^{-1} \\ &\quad = \big(I_1 - J_1 M_{11}^D(t_0+iy)\big)^{-1}\big(J_1 M_{11}^D(t_0+iy) - J_1 M_{11}^D(t_0)\big)\big(I_1 - J_1 M_{11}^D(t_0)\big)^{-1}. \end{aligned}$$

We obtain $\text{o-}\lim_{y\to+0}(I_1 - J_1 M_{11}^D(t+iy))^{-1} = (I_1 - J_1 M_{11}^D(t))^{-1}$ for a.e. $t \in \delta_a$ which yields the existence of $\Xi(t) := \text{o-}\lim_{y\to+0} \Xi(t+iy)$ and proves representation (4.11).

(ii)$_3$ Next we set

$$\Delta(z) := M_{22}^D(z) + M_{21}^D(z)\big(J_1 - M_{11}^D(z)\big)^{-1} M_{12}^D(z), \quad z \in \mathbb{C}_+$$

and show that the function $T_2(\cdot) := (J_2 - \Delta(\cdot))^{-1}$ is $R_{\mathcal{H}_2}$-function.

Clearly, $\Delta(\cdot)$ is holomorphic in $\mathbb{C}_+$ and it acts in a finite dimensional Hilbert space $\mathcal{H}_2$. Since $\det(J_2 - \Delta(\cdot))$ is also holomorphic in $\mathbb{C}_+$, the determinant $\det(J_2 - \Delta(\cdot))$ has only a discrete set of zeros in $\mathbb{C}_+$. Hence the inverse operator $T_2(\cdot) := (J_2 - \Delta(\cdot))^{-1}$ exists for $z \in \Omega \subset \mathbb{C}_+$ where $\mathbb{C}_+ \setminus \Omega$ is at most countable discrete set, that is, $T_2(\cdot)$ is meromorphic in $\mathbb{C}_+$.

As we just mentioned the inverse operator $(J_2 - \Delta(z))^{-1}$ exists for $z \in \Omega \subset \mathbb{C}_+$. Choose any $z \in \Omega$. Then, by the Frobenius formula,

$$T(z) := \big(J - M^D(z)\big)^{-1} = \begin{pmatrix} T_1(z) & \Xi(z) M_{12}^D(z) T_2(z) \\ T_2(z) M_{21}^D(z) \Xi(z) & T_2(z) \end{pmatrix} \tag{4.12}$$

where

$$T_1(z) := \Xi(z) + \Xi(z)M_{12}^D(z)T_2(z)M_{21}^D(z)\Xi(z). \tag{4.13}$$

Hence

$$T_2(z) = P_2T(z) \upharpoonright \mathcal{H}_2, \quad z \in \Omega.$$

Since $T(\cdot)$ is an $R_{\mathcal{H}}$-function, we get that $\mathrm{Im}(T_2(z)) > 0$ for $z \in \Omega$. Since in addition $T_2(\cdot)$ is meromorphic in $\mathbb{C}_+$, we conclude that it is holomorphic. Thus, $T_2(\cdot) = (J_2 - \Delta(\cdot))^{-1}$ is $R_{\mathcal{H}_2}$-function, too.

(ii)$_4$ In this step we show that for any $a > 0$ the limit $T(t) := \text{o-lim}_{y\to+0} T(t+iy)$ exists in the operator norm for a.e. $t \in \delta_a$. Since $T_2(\cdot)$ is the matrix $R_{\mathcal{H}_2}$-function, the limit $T_2(t) = \text{o-lim}_{y\to+0} T_2(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Besides, (4.9) yields

$$\lim_{y\to+0} \left\| M_{12}^D(t+iy) - M_{12}^D(t) \right\| = 0 \quad \text{and} \quad \lim_{y\to+0} \left\| M_{21}^D(t+iy) - M_{21}^D(t) \right\| = 0$$

for a.e. $t \in \mathbb{R}$. Combining these relations with (4.11) and (4.13) yields the existence of the limit $T_1(t) := \text{o-lim}_{y\to+0} T_1(t+iy)$ for a.e. $t \in \delta_a$. Finally, combining all these relations with the block-matrix representation (4.12) we complete the proof of (ii).

(iii) Using the results of (ii) we are now going to complete the proof of the theorem. We set $\delta_n := \{t \in \mathbb{R}\colon m^+(t) \leqslant n\}$ and note that $\bigcup_{n\in\mathbb{N}} \delta_n$ differs from $\mathbb{R}$ by a set of Lebesgue measure zero. By step (ii) the limit $T(t) := \text{o-lim}_{y\to+0} T(t+iy)$ exists for a.e. $t \in \bigcup_{n\in\mathbb{N}} \delta_n$ in the operator norm. Hence the limit $T(t) := \text{o-lim}_{y\to+0} T(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Combining this fact with (4.9) we can pass to the limit in the identity $(J - M^D(t+iy))T(t+iy) = I$ as $y \to 0$. We get

$$\bigl(J - M^D(t)\bigr)T(t) = T(t)\bigl(J - M^D(t)\bigr) = I \quad \text{for a.e. } t \in \mathbb{R}. \tag{4.14}$$

The rest of the proof is similar to that of Theorem 4.1. First we assume that $\widetilde{A}$ is disjoint with A_0, hence, it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$. Therefore, setting $M_B(\cdot) := (B - M(\cdot))^{-1}$ and assuming without loss of generality that $0 \in \varrho(B)$ we arrive at the representation (4.6) with $D = |B|^{-1/2}$ for a.e. $t \in \mathbb{R}$. Moreover, (4.14) yields $\mathrm{ran}(T(t)) = \mathcal{H}$ for a.e. $t \in \mathbb{R}$. Therefore arguing as in (4.7) and (4.8) we obtain

$$\begin{aligned} d_{M_B}(t) &= \dim\Bigl(\mathrm{ran}\Bigl(\sqrt{\mathrm{Im}\bigl(M^D(t)\bigr)}\Bigr)\Bigr) = \dim\Bigl(\mathrm{ran}\Bigl(\sqrt{\mathrm{Im}\bigl(M(t)\bigr)}D\Bigr)\Bigr) \\ &= \dim\Bigl(\mathrm{ran}\Bigl(\sqrt{\mathrm{Im}\bigl(M(t)\bigr)}\Bigr)\Bigr) = \dim\bigl(\mathrm{ran}\bigl(\mathrm{Im}\bigl(M(t)\bigr)\bigr)\bigr) = d_M(t) \end{aligned}$$

for a.e. $t \in \mathbb{R}$. Applying Theorem 3.4(ii) we complete the proof.

Finally, we apply Corollary 4.2 to extend the proof for extensions $\widetilde{A}$ not disjoint with A_0. □

Remark 4.4. Note that in passing we proved the following "individual" version of Theorem 4.3.

If the extension $\widetilde{A} = \widetilde{A}^* = \widetilde{A}_B (\in \mathrm{Ext}_A)$ satisfies conditions (1.1) and (4.9) with $D = |B|^{-1/2}$, then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.

This observation shows that the classical Kato–Rosenblum theorem, as well as its generalization, Theorem 4.1, is implied by Theorem 4.3. Indeed, the condition (4.3) is equivalent to $D \in \mathfrak{S}_2$, hence the limit (4.9) exists even in $\mathfrak{S}_2$-norm (cf. [5,15]).

However, we presented the direct proof of Theorem 4.1 because of its simplicity.

Remark 4.5. Theorem 4.3 as well as its proof remains valid if A is non-densely defined. In this case it suffices to use the boundary triplet technique for non-densely defined operators developed in [12,26], cf. proof of Theorem 4.1. However, the assumptions on the Weyl function are indispensable.

The following local version of Theorem 4.3 is implied by combining Theorem 3.4(ii) with the proof of Theorem 4.3.

Corollary 4.6. *Let the assumptions of Theorem* 4.3 *be satisfied and let*

$$\mathcal{F} := \{t \in \mathbb{R}: \ m^+(t) < \infty\}. \tag{4.15}$$

If condition (1.1) *holds, then the parts* $\widetilde{A}^{ac} E_{\widetilde{A}^{ac}}(\mathcal{F})$ *and* $A_0^{ac} E_{A_0^{ac}}(\mathcal{F})$ *of* $\widetilde{A}$ *and* A_0*, respectively, are unitarily equivalent.*

Remark 4.7. Let us define the invariant maximal normal function

$$\mathfrak{m}^+(t) := \sup_{y \in (0,1]} \left\| \operatorname{Im}\big(M(i)\big)^{-1/2}\big(M(t+iy) - \operatorname{Re}\big(M(i)\big)\big)\operatorname{Im}\big(M(i)\big)^{-1/2} \right\|, \tag{4.16}$$

for $t \in \mathbb{R}$. For Weyl functions one easily proves that $m^+(t)$ is finite if and only if $\mathfrak{m}^+(t)$ is finite.

(i) The quantity $\mathfrak{m}^+(t)$ has the advantage that it is invariant: Let A be a densely defined closed symmetric operator, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triplet for A^*, and $M(\cdot)$ the corresponding Weyl function. Further, let $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be another boundary triplet for A^* with the Weyl function $\widetilde{M}(\cdot)$ and let $A_0 := A^* \restriction \ker(\Gamma_0) = A^* \restriction \ker(\widetilde{\Gamma}_0)$. In this case $M(\cdot)$ and $\widetilde{M}(\cdot)$ are related by (2.12) However, $\widetilde{\mathfrak{m}}^+(t) = \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$, where $\mathfrak{m}^+(t)$ is obtained by replacing in (4.16) $M(\cdot)$ by $\widetilde{M}(\cdot)$.

(ii) Further, if the Weyl function $M(\cdot)$ satisfies $M(i) = i$, then $m^+(t) = \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$.

(iii) Let π be an orthogonal projection onto a subspace $\widehat{\mathcal{H}}$ of $\mathcal{H}$. If $\mathfrak{m}^+(t)$ is finite, then the invariant maximal normal function $\widehat{\mathfrak{m}}^+(t)$, obtained from (4.16) replacing $M(\cdot)$ by $\widehat{M}(\cdot) := \pi M(\cdot) \restriction \widehat{\mathcal{H}}$, is also finite and satisfies $\widehat{\mathfrak{m}}^+(t) \leqslant \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$.

5. Concluding remarks

Here we demonstrate that condition (1.3) might have much stronger conclusions than Theorem 1.2. For this purpose we complete Definition 3.5.

Definition 5.1. Let A be a symmetric operator in $\mathfrak{H}$, $A_0 = A_0^* \in \mathrm{Ext}_A$ and $\sigma_0 := \sigma_{ac}(A_0)$. We say that A_0 is *strictly ac-minimal* if for any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ the *ac*-part $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma_0)$ of $\widetilde{A} E_{\widetilde{A}}(\sigma_0)$ is unitarily equivalent to A_0^{ac}.

In [28] we applied Theorem 1.2 as well as technique elaborated in this paper to direct sums $A := \bigoplus_{n=1}^{\infty} S_n$ of closed symmetric operators S_n with finite deficiency indices. It turns out that in this case for a suitable boundary triplet Π for A^* the corresponding extension A_0 is *ac-minimal* provided that condition (1.3) is satisfied, cf. [28, Theorem 5.12]. Moreover, if the symmetric operators S_n are mutually unitary equivalent, then for a suitable boundary triplet Π for A^* the extension A_0 is actually *strictly ac-minimal*.

Moreover, in [28] the Sturm–Liouville operator

$$(Af)(x) = -f''(x) + Tf(x)$$

with non-negative unbounded operator potential T was considered. It is shown in [28] that condition (1.3) is satisfied for the Weyl function of the pair $\{A, A_F\}$ and, by [28, Theorem 6.11(ii)], the Friedrichs extension $A^F =: A_0$ is *ac-minimal*. In particular, this yields $\sigma_{ac}(A^F) \subseteq \sigma_{ac}(\widetilde{A})$ for any $\widetilde{A} \in \mathrm{Ext}_A$, i.e. $\sigma_{ac}(A^F)$ is stable under non-additive perturbations preserving the class Ext_A. In this case the inequality $N_{\widetilde{A}^{ac}}(t) \geqslant N_{A_0^{ac}}(t)$ holds for the spectral multiplicity functions. Moreover, if $\inf \sigma_{\mathrm{ess}}(T) = \inf \sigma(T)$, then both A^F and the Krein extension A^K are *strictly ac-minimal*, cf. [28, Corollary 6.12].

Finally, in [28] we apply the above mentioned results for the investigation of self-adjoint realizations of partial differential expressions of the form

$$\mathcal{L} = -\left(\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right) + q(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ 0 \leqslant q = \bar{q} \in L^\infty\big(\mathbb{R}^n\big),$$

in the half-space $\mathbb{R}_+ \times \mathbb{R}^n$. Let $L := L_{\min}$ be the minimal symmetric operator associated with the differential expression $\mathcal{L}$ in $\mathfrak{H} := L^2(\mathbb{R}_+ \times \mathbb{R}^n)$.

Denote also by L^D, L^N and L^K the Dirichlet, Neumann and Krein realizations of $\mathcal{L}$ (extensions of L), respectively. Note that the realizations L^D and L^N are always self-adjoint (cf. [25, Theorem 2.8.1], [18]).

Theorem 5.2. *Let* $q(\cdot) \in L^\infty(\mathbb{R})$, $q(\cdot) \geqslant 0$, *and*

$$\lim_{|x| \to \infty} \int_{|x-y| \leqslant 1} \big|q(y)\big|\, dy = 0.$$

Let also $\widetilde{L} = \widetilde{L}^* \in \mathrm{Ext}_L$. *Then*:

(i) *The realizations* L^D *and* L^N *are absolutely continuous,* $L^D = (L^D)^{ac}$ *and* $L^N = (L^N)^{ac}$.
(ii) $\widetilde{L}^{ac}$ *and* L^D *are unitarily equivalent provided that either the condition* $(\widetilde{L} - i)^{-1} - (L^D - i)^{-1} \in \mathfrak{S}_\infty$ *or* $(\widetilde{L} - i)^{-1} - (L^K - i)^{-1} \in \mathfrak{S}_\infty$ *is satisfied.*
(iii) *The realizations* L^D, L^N *and* L^K *are strictly ac-minimal,*

$$\sigma\big(L^D\big) = \sigma_{ac}\big(L^D\big) = \sigma_{ac}\big(L^K\big) = \sigma\big(L^N\big) = \sigma_{ac}\big(L^N\big) = [0, \infty),$$

and $N_{L^D}(t) = N_{L^N}(t) = N_{(L^K)^{ac}}(t) = \infty$ *for a.e.* $t \in [0, \infty)$.

The proof is contained in our preprint [28]. Note only that, since condition (1.3) is now satisfied, the statement (ii) follows from Theorem 1.2.

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Appendix A. Absolutely continuous closure

The concept of the *ac*-closure has been introduced in [8] (see also [14]). Its properties can also be found in [8,14]. Here we recall some basic facts on the *ac*-closure of a Borel subset of $\mathbb{R}$ that were used in Section 3.

Definition A.1. (See [8].) Let $\delta \in \mathcal{B}(\mathbb{R})$. The set $\mathrm{cl}_{ac}(\delta)$ defined by

$$\mathrm{cl}_{ac}(\delta) := \{x \in \mathbb{R}: \ |(x-\varepsilon, x+\varepsilon) \cap \delta| > 0 \ \forall \varepsilon > 0\}$$

is called the absolutely continuous closure of the Borel set $\delta \in \mathcal{B}(\mathbb{R})$.

Obviously, two Borel sets $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$ have the same *ac*-closure if their symmetric difference $\delta_1 \vartriangle \delta_2$ has Lebesgue measure zero. Moreover, the set $\mathrm{cl}_{ac}(\delta)$ is always closed and $\mathrm{cl}_{ac}(\delta) \subseteq \overline{\delta}$. In particular, if we have two measurable non-negative functions ξ_1 and ξ_2 which differ only on a set of Lebesgue measure zero, then $\mathrm{cl}_{ac}(\mathrm{supp}(\xi_1)) = \mathrm{cl}_{ac}(\mathrm{supp}(\xi_2))$.

Lemma A.2. *If $\delta \in \mathcal{B}(\mathbb{R})$, then $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$.*

Proof. Since $\mathrm{cl}_{ac}(\delta)$ is closed the set $\Delta := \mathbb{R} \setminus \mathrm{cl}_{ac}(\delta)$ is open. The open set Δ is decomposed as $\Delta = \bigcup_{l=1}^{L} \Delta_l$, $1 \leqslant L \leqslant \infty$, where $\Delta_l = (a_l, b_l)$ are disjoint open intervals. We set $\Delta_l = \delta \cap \Delta_l$, $l = 1, 2, \ldots, L$. Obviously,

$$\delta \setminus \mathrm{cl}_{ac}(\delta) = \delta \cap \Delta = \bigcup_{l=1}^{L} \Delta_l.$$

We note that $\Delta_l \cap \mathrm{cl}_{ac}(\delta) = \emptyset$, $l = 1, 2, \ldots, L$. Hence for each $t \in \Delta_l$ there is a sufficiently small neighborhood $\mathcal{O}_t$ such that $|\mathcal{O}_t \cap \delta| = 0$. If η is sufficiently small, then $[a_l + \eta, a_l - \eta] \subseteq (a_l, b_l)$ and $\{\mathcal{O}_t\}_{t \in \Delta_l}$ forms a covering of $[a_l + \eta, a_l - \eta]$. Since $[a_l + \eta, a_l - \eta]$ is compact we can chosen a finite covering $\{\mathcal{O}_{t_m}\}_{m=1}^{M}$ of $[a_l + \eta, a_l - \eta]$. By $[a_l + \eta, a_l - \eta] \subseteq \bigcup_{m=1}^{M} \mathcal{O}_{t_m}$ we find $|[a_l + \eta, a_l - \eta] \cap \delta| = 0$ for each sufficiently small $\eta > 0$. Using that we get

$$\begin{aligned}|(a_l, b_l) \cap \delta| &= |(a_l, a_l + \eta) \cap \delta| + |(b_l - \eta, b_l) \cap \delta| \\ &= |(a_l, a_l + \eta) \cap \delta| + |(b_l - \eta, b_l) \cap \delta| \leqslant 2\eta\end{aligned}$$

for sufficiently small $\eta > 0$. Hence $|\Delta_l| = |(a_l, b_l) \cap \delta| = 0$ which yields that $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$. □

Lemma A.3. *If $\{\delta_k\}_{k \in \mathbb{N}}$, $\delta_k \subseteq \mathbb{R}$, is a sequence of Borel subsets, then*

$$\mathrm{cl}_{ac}(\delta) = \overline{\bigcup_{k \in \mathbb{N}} \mathrm{cl}_{ac}(\delta_k)}, \qquad \delta = \bigcup_{k \in \mathbb{N}} \delta_k. \tag{A.1}$$

Proof. We set $\widehat{\delta_k} = \delta_k \cap \mathrm{cl}_{ac}(\delta_k)$ and $\Delta_k := \delta_k \setminus \mathrm{cl}_{ac}(\delta_k)$. We have $\delta = \widehat{\delta} \cup \Delta$, where $\widehat{\delta} := \bigcup_{k\in\mathbb{N}} \widehat{\delta_k}$ and $\Delta := \bigcup_{k\in\mathbb{N}} \Delta_k$. By Lemma A.2, $|\Delta_k| = 0$, $k \in \mathbb{N}$, which yields $|\Delta| = 0$. Hence $\mathrm{cl}_{ac}(\delta) = \mathrm{cl}_{ac}(\widehat{\delta})$. Similarly one gets $\mathrm{cl}_{ac}(\delta_k) = \mathrm{cl}_{ac}(\widehat{\delta_k})$, $k \in \mathbb{N}$. Notice that $\widehat{\delta_k} \subseteq \mathrm{cl}_{ac}(\widehat{\delta_k})$, $k \in \mathbb{N}$. We have

$$\mathrm{cl}_{ac}(\widehat{\delta}) \supseteq \bigcup_{k\in\mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta_k}) \supseteq \bigcup_{k\in\mathbb{N}} \widehat{\delta_k} = \widehat{\delta}.$$

Hence

$$\mathrm{cl}_{ac}(\widehat{\delta}) = \overline{\mathrm{cl}_{ac}(\widehat{\delta})} \supseteq \overline{\bigcup_{k\in\mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta_k})} \supseteq \bar{\widehat{\delta}} \supseteq \mathrm{cl}_{ac}(\widehat{\delta})$$

which yields $\mathrm{cl}_{ac}(\widehat{\delta}) = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta_k})}$. Since $\mathrm{cl}_{ac}(\widehat{\delta}) = \mathrm{cl}_{ac}(\delta)$ and $\mathrm{cl}_{ac}(\widehat{\delta_k}) = \mathrm{cl}_{ac}(\delta_k)$, $k \in \mathbb{N}$, we prove (A.1). $\square$

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